UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1403**
ASSESSMENT PATTERN	:	**MATH1403A**
MODULE NAME	:	**Mathematical Methods for Arts and Sciences**
DATE	:	**N/A (Practice Paper A)**
TIME	:	**N/A**
TIME ALLOWED	:	**2 Hours 0 Minutes**

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

   (b) Use the definition of $\sinh(x)$ to show that

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{1+x^2}\right).$$

   (c) Hence find expressions for the first and second derivatives of $\sinh^{-1}(x)$.

   (d) The curve $y(x)$ is defined by

$$\sinh[y(x)] = x^2 - 1.$$

   Find:

   (i) the values of x where $y(x)$ cuts the x-axis,

   (ii) the values of x at any stationary points of $y(x)$,

   (iii) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

2. (a) (i) Find (or state) the first three terms of the Maclaurin series for the functions e^x and $\sin x$.

(ii) Hence find the first three terms of the Maclaurin expansion of $e^{\sin x}$.

(iii) Hence show that

$$\int_0^1 e^{\sin x}\,\mathrm{d}x \approx \frac{5}{3}.$$

(The actual answer can be worked out numerically as $1.63187\ldots$.)

(b) The *Euler–Tricomi equation* is a partial differential equation which is useful in the study of extremely fast air flows (in the range 600–768 mph), given by

$$\frac{\partial^2 u}{\partial x^2} - x\frac{\partial^2 u}{\partial y^2} = 0.$$

Which of these functions $u(x, y)$ below satisfy the Euler–Tricomi equation?

(i) $u(x, y) = 2(3y^2 + x^3) - (y^3 + x^3 y)$

(ii) $u(x, y) = x\cos y + 1$

3. (a) Evaluate the integrals:

(i) $\displaystyle\int_0^1 \sqrt{\frac{x+2}{2}}\,\mathrm{d}x$,

(ii) $\displaystyle\int \ln(\ln x)\frac{\ln x}{x}\,\mathrm{d}x$,

(iii) $\displaystyle\int_{-\pi/2}^{0} \sin^3 x\cos^3 x\,\mathrm{d}x$.

(b) Show that

$$\int_{-\pi}^{\pi} e^x\sin(3x)\,\mathrm{d}x = \frac{3}{5}\sinh(\pi).$$

4. (a) Let $z = \sqrt{2} - 2i$ and $w = -\sqrt{3} - i$. Write down:

   (i) $\mathrm{Re}(iw)$,

   (ii) z/w,

   (iii) $|\overline{zw}|$,

   (iv) $\arg(w)$,

   writing your answers in the form $x + iy$ where appropriate.

   (b) (i) Find all solutions to $z^4 = 1 + i$ in the form $r(\cos\theta + i\sin\theta)$ where $r, \theta \in \mathbb{R}$ and $r > 0$.

   (ii) Plot these solutions on an Argand diagram.

   (c) State DeMoivre's theorem and use it to express $\cos 6\theta$ as a polynomial in $\cos\theta$.

5. Solve the following ordinary differential equations:

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x(y^2 + 1), \quad y(0) = 1,$

   (b) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = x\sin x,$

   (c) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = \dfrac{3}{x^2}.$

6. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (b) Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x \leqslant 0 \\ -2\pi x & \text{if } 0 < x < \pi \end{cases},$$

   where $f(x + 2\pi) = f(x)$.

   (c) Using part (b), or otherwise, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$