UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1403**
ASSESSMENT PATTERN	:	**MATH1403A**
MODULE NAME	:	**Mathematical Methods for Arts and Sciences**
DATE	:	**N/A (Practice Paper B)**
TIME	:	**N/A**
TIME ALLOWED	:	**2 Hours 0 Minutes**

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All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) (i) By writing $\tan x = \dfrac{\sin x}{\cos x}$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan x\right] = \sec^2 x.$$

(ii) Use this in showing that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] = \frac{1}{a^2+x^2}.$$

(b) The curve $y(x)$ is defined on $[-\pi, \pi]$ by

$$y(x) = \frac{1}{2}\left[2\cos^4 x + \sin^2 x - 1\right].$$

Find:

(i) the values of x where $y(x)$ cuts the x-axis,
(ii) the values of x at any stationary points of $y(x)$,
(iii) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

2. (a) Find (or state) the first three non-zero terms of the Maclaurin series for the function $f(x) = \dfrac{1}{x-1}$ (where $|x| < 1$).

(b) Find the first four terms of a Taylor series solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^2 + y^3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2,$$

where $y(1) = 1$ and $y'(1) = 1$.

(c) Find the equation of the plane tangent to the curve

$$z(x, y) = \frac{\cos(3x - y)}{2y},$$

at the point $\left(\frac{\pi}{3}, \frac{\pi}{2}, 0\right)$.

3. (a) Evaluate the integrals:

(i) $\displaystyle\int \left(\frac{x}{5} - \pi\right)^{1/3} \mathrm{d}x$,

(ii) $\displaystyle\int x\sec^2 x \,\mathrm{d}x$,

(iii) $\displaystyle\int_0^\infty \frac{e^x}{(1+e^x)^2}\,\mathrm{d}x$.

(b) Show that

$$\int_{-1/2}^{1/2} \frac{x^4+1}{x^3-1}\,\mathrm{d}x = -\frac{1}{3}\ln(21).$$

Hint: recall that $x^3 - 1 = (x-1)(x^2+x+1)$.

4. (a) (i) For a complex number z, define its *complex conjugate*, *modulus*, *argument* and *principal argument*.

   (ii) Define the complex logarithm $\ln(z)$ and the principal-valued complex logarithm $\mathrm{Ln}(z)$ in terms of the modulus, argument and/or principal argument of z.

   (iii) Find **all** solutions to the equation $e^{2z} = 1 - i$.

   (b) Let $z = \cos\theta + i \sin\theta$.

   (i) State De Moivre's theorem.

   (ii) Show that $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$.

   (iii) Use this to show that

$$\sin^3\theta \cos^3\theta = \frac{1}{32}\left[3\sin(2\theta) - \sin(6\theta)\right].$$

   (iv) Hence, or otherwise, show that

$$\int_0^{\pi/2} \cos^3 x \sin^3 x \, \mathrm{d}x = \frac{1}{12}.$$

5. (a) Find the solution to the first order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-x}.$$

   (b) By choosing a suitable substitution, show that the solution to

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 + x^2 = 0, \quad y(1) = 0$$

   can be written as

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2,$$

   (i.e. a circle of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0)$.)

   (c) Solve the second order ordinary differential equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 10\sinh x.$$

6. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (b) Find the Fourier series for

$$f(x) = x(\pi - x), \quad -\pi < x < \pi,$$

   where $f(x + 2\pi) = f(x)$.

   (c) Hence, or otherwise, find the Fourier series of $g(x) = x(\pi - 2x)$ on $(-\pi, \pi)$.