UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1403**
ASSESSMENT PATTERN	:	**MATH1403A**
MODULE NAME	:	**Mathematical Methods for Arts and Sciences**
DATE	:	**07-Jan-13**
TIME	:	**09:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) The quotient rule for differentiating $f(x) = \dfrac{u(x)}{v(x)}$ with respect to x is given by

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}.$$

By writing $f(x) = u(x) \cdot \dfrac{1}{v(x)}$, derive the quotient rule using the product rule and the chain rule.

(b) (i) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}.$$

(ii) The curve $y(x)$ is defined in the region $(-1, 1)$ by

$$y(x) = -\arcsin\left(x^2\right) + \frac{\pi}{6}.$$

Find:

(A) the values of x where $y(x)$ cuts the x-axis,
(B) the values of x at any stationary points of $y(x)$,
(C) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

2. (a) By writing $\operatorname{cosec} x = \dfrac{1}{\sin x}$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosec} x\right] = -\operatorname{cosec} x \cot x.$$

(b) Find the first four terms of a Taylor series solution for the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\operatorname{cosec} y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 1 = 0.$$

where $y'(0) = 1/2$ and $y(0) = \pi/2$.

(c) The *wave equation* is a partial differential equation which describes the movement of waves over time, given for spherical waves by

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = 0.$$

If $c = 2$, which of these functions $u(r,t)$ below satisfy the wave equation?

(i) $u(r,t) = \dfrac{2\sin(r)\cos(2t)}{r}$

(ii) $u(r,t) = e^{-t}\cos(r)$

3. (a) Evaluate the integrals:

(i) $\displaystyle\int \left(\frac{1}{3}x - 1\right)^{-1/2} \mathrm{d}x,$

(ii) $\displaystyle\int_{\sqrt{3}}^{1} \frac{2x-3}{x^3+x}\,\mathrm{d}x,$

(iii) $\displaystyle\int_0^{\infty} x e^{-x}\,\mathrm{d}x.$

(b) Use integration by parts to show that

$$\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}\left[\ln x\right]^2 + c,$$

where c is a constant.

4. (a) Let $z = -\sqrt{2} + 2i$ and $w = -1 + i$. Write down:
   (i) $z/\overline{w}$,
   (ii) $\arg(w)$,

   where your answer for (i) is in the form $x + iy$, where $x, y \in \mathbb{R}$.

   (b) (i) Find **all** solutions to the equation $e^z = 1 + i$.
   (ii) Plot these solutions on an Argand diagram.

   (c) Let $z = \cos\theta + i\sin\theta$.
   (i) Show that $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$.
   (ii) Show by expanding $(z + z^{-1})^5$ that

$$\cos^5\theta = \frac{1}{16}\left[\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta)\right].$$

5. (a) Solve the first order ordinary differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2 \ln x.$$

   Hint: You can use the result of question 3b without proof.

   (b) By choosing a suitable substitution, show that the solutions to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{x}{y}$$

   are given by

$$y(x) = \pm x\sqrt{2\ln x + c},$$

   where c is a constant.

   (c) Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sinh x \sinh 2x,$$

   where $y(0) = 1$ and $y'(0) = 0$.

6. (a) Define $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

(b) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

(c) Find the Fourier series for

$$f(x) = e^x, \quad -\pi < x < \pi$$

where $f(x + 2\pi) = f(x)$.

(d) Hence, or otherwise, find the Fourier series of $\sinh x$ and $\cosh x$ on $(-\pi, \pi)$.