UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1403**
ASSESSMENT PATTERN	:	**MATH1403A**
MODULE NAME	:	**Mathematical Methods for Arts and Sciences**
DATE	:	**23-May-13**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2012/13-MATH1403A-001-EXAM-29

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[2^x\right]$.

   (b) (i) Define $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

   (ii) Use these definitions of $\cosh(x)$ and $\sinh(x)$ to:

   (A) find their derivatives,

   (B) obtain the formula $\cosh^2 x - \sinh^2 x = 1$.

   (iii) The curve $y(x)$ is defined by

$$y(x) = \sinh^2 x - \cosh x - 1.$$

   Find:

   (A) the two real values of x where $y(x)$ cuts the x-axis,

   (B) the values of x at any stationary points of $y(x)$,

   (C) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

2. (a) By writing $\sec x = \dfrac{1}{\cos x}$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x.$$

(b) (i) Find the first two non-zero terms in the Maclaurin series of

$$f(x) = \ln(\cos x).$$

(ii) Hence, or otherwise, given that $\cos^2\left(\dfrac{x}{2}\right) = \dfrac{1+\cos x}{2}$, show that

$$\ln(1+\cos x) = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} + \cdots .$$

(c) The *heat equation* is a partial differential equation which describes the distribution of heat in a given region over time, given by

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

If $\alpha = 1$, determine whether the following functions $u(x, y, t)$ satisfy the heat equation:

(i) $u(x, y, t) = t\cos(x)\sin(y)$,
(ii) $u(x, y, t) = e^{-t}(\cos(y) + \sin(x))$.

3. (a) Evaluate the integrals:

(i) $\displaystyle\int \left(2x + \frac{1}{2}\right)^{3/2} \mathrm{d}x$,
(ii) $\displaystyle\int_1^3 \frac{x^3+2}{x(x+1)}\,\mathrm{d}x$.

(b) Show that

$$\int_{1/2}^{1} \frac{1}{x(5x^2-4x+1)^{1/2}}\,\mathrm{d}x = \sinh^{-1} 1.$$

Hint: use $x = 1/u$.

4. (a) Let $z = 3 + 2i$ and $w = -1 - i$. Write down:

   (i) $\mathrm{Im}(w)$,

   (ii) zw,

   (iii) $z/\overline{w}$,

   (iv) $\arg(w)$,

   where your answers for (ii) and (iii) are in the form $x + iy$.

   (b) (i) Find all solutions to $z^5 = -32$ in the form $r(\cos\theta + i\sin\theta)$ where $r, \theta \in \mathbb{R}$ and $r > 0$.

   (ii) Plot these solutions on an Argand diagram.

   (c) Let $z = \cos\theta + i\sin\theta$.

   (i) Show that $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$.

   (ii) Show by expanding $(z - z^{-1})^5$ that

$$\sin^5\theta = \frac{1}{16}\left[\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)\right].$$

5. Solve the following ordinary differential equations:

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\tan y}{x}$,

   (b) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^3\ln x, \quad y(1) = -1$,

   (c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x + 3x + 2$.

6. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (b) Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x \leqslant -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} < x \leqslant \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x < \pi \end{cases},$$

   where $f(x + 2\pi) = f(x)$.

   (c) Using part (b), or otherwise, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$