UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1403**
ASSESSMENT PATTERN	:	**MATH1403A**
MODULE NAME	:	**Mathematical Methods for Arts and Sciences**
DATE	:	**13-Jan-14**
TIME	:	**08:45**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2013/14-MATH1403-001-EXAM-J1

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) By applying the product rule twice, show that for three functions $u(x)$, $v(x)$, $w(x)$,
$$(uvw)' = u'vw + uv'w + uvw',$$
where dashes represent differentiation with respect to x.
   (b) Define $\sinh(x)$ and $\cosh(x)$ in terms of exponentials. Hence find their derivatives.
   (c) Define $\tanh(x)$ and $\text{sech}(x)$ in terms of $\sinh(x)$ and $\cosh(x)$. Hence find their derivatives.
   (d) The curve $y(x)$ is given by
$$y(x) = \sinh(\text{sech}(x)) + \sinh(-1).$$
Find:
      (i) the values of x where $y(x)$ cuts the x-axis,
      (ii) the values of x at any stationary points of $y(x)$,
      (iii) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

2. (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}.$$

(b) The curve $y(x)$ is given by

$$y(x) = \frac{1}{\tan^{-1}(1+2x)}.$$

(i) Show that

$$(1+2x+2x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0.$$

(ii) Hence find the first three nonzero terms of a Maclaurin series for $y(x)$.

(c) Find the equation of the plane tangent to the curve

$$z(x,y) = \frac{\sin(y)}{\sqrt{x+y}},$$

at the point $\left(0, \frac{\pi}{3}, \frac{3}{2\sqrt{\pi}}\right)$.

3. (a) Evaluate the integrals:

(i) $\displaystyle\int \left(\frac{x+1}{5}\right)^{-1/5} \mathrm{d}x$,

(ii) $\displaystyle\int_{-\infty}^{0} (x+1)e^x \, \mathrm{d}x$,

(iii) $\displaystyle\int \frac{3x+2}{x^2+4x-5} \, \mathrm{d}x$.

(b) Evaluate

$$\int_0^1 \frac{x^2}{(1+x^2)^2} \, \mathrm{d}x.$$

Hint: start with the substitution $x = \tan\theta$.

4. (a) For a complex number z, define its *complex conjugate*, *modulus*, *argument* and *principal argument*.

   (b) (i) Write down the modulus and principal argument of:

   (A) $z_1 = 2 + 2i$,

   (B) $z_2 = -3\sqrt{3} - 3i$.

   (ii) Let $w = z_1/z_2$. Use part (i) to write w and $\overline{w}$ in the form $r(\cos\theta + i\sin\theta)$, where $r, \theta \in \mathbb{R}$, $r > 0$ and $-\pi < \theta \leqslant \pi$.

   (iii) State De Moivre's theorem.

   (iv) Hence write w^{12} in modulus–argument form, and show that it is real.

   (c) (i) Define the complex logarithm $\ln(z)$ and the principal-valued complex logarithm $\mathrm{Ln}(z)$ in terms of the modulus, argument and/or principal argument of z.

   (ii) Find **all** solutions to the equation $2e^{3x} = -6\sqrt{3} - 6i$.
   Hint: you might want to use your answers to part (b)(i)(B).

   (iii) Plot these solutions on an Argand diagram.

5. Solve the differential equations:

   (a) $\dfrac{1}{x+2}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x}$,

   (b) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = -x, \quad y'(1) = 0$,

   (c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = x^2 + 2\sin x$.

6. (a) What does it mean for a function to be

   (i) odd?

   (ii) even?

   (b) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (c) Find the Fourier series for

$$f(x) = \cosh(x),$$

   where $f(x + 2\pi) = f(x)$.

   (d) Using part (c), or otherwise, show that

$$\frac{1}{2}[\pi\coth(\pi) - 1] = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots .$$