MATH1403 EXAM JANUARY 2014 SOLUTIONS

1.(a) $$(uvw)' = ([uv]w)'$$

Product rule $(uv)' = u'v + uv'$

$$= [uv]'w + [uv]w'$$
$$= [u'v + uv']w + [uv]w'$$
$$= \underline{u'vw + uv'w + uvw'}.$$

(b) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\Rightarrow \frac{d}{dx}\left[\sinh(x)\right] = \frac{d}{dx}\left[\tfrac{1}{2}(e^x - e^{-x})\right]$$
$$= \tfrac{1}{2}\left[e^x - (-e^{-x})\right]$$
$$= \tfrac{1}{2}(e^x + e^{-x}) = \underline{\cosh(x)}$$

$$\frac{d}{dx}\left[\cosh(x)\right] = \frac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right]$$
$$= \tfrac{1}{2}\left[e^x - e^{-x}\right] = \underline{\sinh(x)}.$$

(c) $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$, $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$

$$\Rightarrow \frac{d}{dx}\left[\tanh(x)\right] = \frac{d}{dx}\left[\frac{\sinh(x)}{\cosh(x)}\right]$$

Quotient rule $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

$$= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)}$$
$$= \frac{1}{\cosh^2(x)} = \underline{\operatorname{sech}^2 x}$$

$$\frac{d}{dx}\left[\mathrm{sech}(x)\right] = \frac{d}{dx}\left[\frac{1}{\cosh x}\right]$$

$$= \frac{-1 \cdot \sinh x}{\cosh^2 x} = \underline{-\mathrm{sech}\,x \tanh x}.$$

(d) $y(x) = \sinh[\mathrm{sech}(x)] + \sinh(-1)$.

(i) Cuts x-axis when $y = 0 \Rightarrow$

$$0 = \sinh[\mathrm{sech}\,x] + \sinh(-1)$$

$$\Rightarrow \quad \sinh[\mathrm{sech}\,x] = -\sinh(-1)$$

$$= \sinh(1)$$

sinh is an odd fn.

$$\Rightarrow \quad \mathrm{sech}\,x = 1$$

arsinh-ing both sides

$$\Rightarrow \quad \cosh x = 1$$

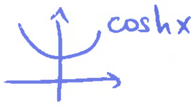
$$\Rightarrow \quad \underline{x = 0}$$

(ii) Stationary points where $y'(x) = 0$

$$y'(x) = -\cosh[\mathrm{sech}(x)]\,\mathrm{sech}\,x \tanh x$$

$$y'(x) = 0 \Rightarrow \quad 0 = -\cosh[\mathrm{sech}\,x]\,\mathrm{sech}\,x \tanh x$$

$\Rightarrow$ either $\cosh[\mathrm{sech}\,x] = 0$
or $\mathrm{sech}\,x = 0$ } impossible



or $\tanh x = 0 \Rightarrow \underline{x = 0}$

(iii) Nature: need $y''(x)$ at the stationary point.

$$y''(x) = \sinh[\mathrm{sech}\,x]\,\mathrm{sech}^2 x \tanh^2 x + \cosh[\mathrm{sech}(x)]\,\mathrm{sech}\,x \tanh^2 x - \cosh[\mathrm{sech}\,x]\,\mathrm{sech}^3 x$$

$$y''(0) = 0 + 0 - \cosh(1) < 0 \Rightarrow \underline{\text{maximum}}$$

2. (a) $y = \tan^{-1} x$. Want $\frac{dy}{dx}$.

$$\Rightarrow \tan y = x$$

diff. $\Rightarrow \sec^2 y \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

$\sin^2 y + \cos^2 y = 1$

$\Rightarrow \tan^2 y + 1 = \sec^2 y$

$$= \frac{1}{1+\tan^2 y}$$

$$= \underbrace{\frac{1}{1+x^2}}.$$

(b)(i) $y = \dfrac{1}{\tan^{-1}(1+2x)} = [\tan^{-1}(1+2x)]^{-1}$

$$\Rightarrow \frac{dy}{dx} = -[\tan^{-1}(1+2x)]^{-2} \frac{1}{1+(1+2x)^2} 2$$

$$= \frac{-2y^2}{1+(1+2x)^2}$$

$$= \frac{-2y^2}{1+1+4x+4x^2} \quad \cdots\cdots\cdots \quad (*)$$

$$\Rightarrow (2+4x+4x^2)\frac{dy}{dx} + 2y^2 = 0$$

$$\Rightarrow \underbrace{(1+2x+2x^2)\frac{dy}{dx} + y^2 = 0} \quad \cdots\cdots\cdots \quad (+)$$

(ii) To find the Maclaurin series, we need to fill in

$y(x) = y(0) + xy'(0) + \frac{1}{2}x^2 y''(0) + \cdots$

We can see $y(0) = \dfrac{1}{\tan^{-1}(1)} = \dfrac{1}{\pi/4} = \underline{\dfrac{4}{\pi}}$

and $(*)$ tells us $y'(0) = \dfrac{-2\left(\frac{4}{\pi}\right)^2}{2} = \underline{-\dfrac{16}{\pi^2}}$.

To find $y''(0)$, diff. $(+)$:

$$(2+4x)\frac{dy}{dx} + (1+2x+2x^2)\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0$$

at $x=0$: $\quad 2\left(-\frac{16}{\pi^2}\right) + \frac{d^2y}{dx^2}(0) + 2\left(\frac{4}{\pi}\right)\left(-\frac{16}{\pi^2}\right) = 0$

$$\Rightarrow \quad -\frac{32}{\pi^2} + y''(0) - \frac{128}{\pi^3} = 0$$

$$\Rightarrow \quad y''(0) = \frac{32}{\pi^2} + \frac{128}{\pi^3}.$$

$\Rightarrow$ series is $\quad y(x) = \frac{4}{\pi} + \left(-\frac{16}{\pi^2}\right)x + \frac{1}{2}x^2\left(\frac{32}{\pi^2} + \frac{128}{\pi^3}\right) + \cdots$

$$= \frac{4}{\pi} - \frac{16}{\pi^2}x + \frac{16\pi+64}{\pi^3}x^2 + \cdots$$

(c). $z = \frac{\sin(y)}{\sqrt{x+y}}$. Find plane tangent at $\left(0, \frac{\pi}{3}, \frac{3}{\sqrt{\pi}\,2}\right)$, i.e. sub into

$$z - z_0 = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0,z_0)}(x-x_0) + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0,z_0)}(y-y_0)$$

where $(x_0, y_0, z_0) = \left(0, \frac{\pi}{3}, \frac{3}{2\sqrt{\pi}}\right)$.

$$z = \frac{\sin y}{\sqrt{x+y}} = \sin y \cdot (x+y)^{-1/2}$$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\frac{\partial z}{\partial x} = -\frac{1}{2}\sin y\,(x+y)^{-3/2}$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(0,\frac{\pi}{3},\frac{3}{2\sqrt{\pi}})} = -\frac{1}{2}\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}\right)^{-3/2} = -\frac{\sqrt{3}}{4}\left(\frac{3}{\pi}\right)^{3/2} = -\frac{9}{4\pi^{3/2}}.$$

and $\frac{\partial z}{\partial y} = \cos y\,(x+y)^{-1/2} - \frac{1}{2}\sin y\,(x+y)^{-3/2}$

$$\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(0,\frac{\pi}{3},\frac{3}{2\sqrt{\pi}})} = \frac{1}{2}\left(\frac{\pi}{3}\right)^{-1/2} - \frac{9}{4\pi^{3/2}}$$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$$= \frac{\sqrt{3}}{2\pi^{1/2}} - \frac{9}{4\pi^{3/2}}$$

$$\Rightarrow \quad z - \frac{3}{2\sqrt{\pi}} = -\frac{9}{4\pi^{3/2}}x + \left(\frac{\sqrt{3}}{2\pi^{1/2}} - \frac{9}{4\pi^{3/2}}\right)\left(y - \frac{\pi}{3}\right).$$

3 (a)(i) $\int \left(\frac{x+1}{5}\right)^{-1/5} dx = \int \left(\frac{1}{5}x+\frac{1}{5}\right)^{-1/5} dx$

$$= \frac{5}{4}\cdot 5\left(\frac{1}{5}x+\frac{1}{5}\right)^{4/5} + c$$

$$= \frac{25}{4}\left(\frac{1}{5}x+\frac{1}{5}\right)^{4/5} + c$$

(ii) $\int_{-\infty}^{0} (x+1)e^x \, dx = \left[(x+1)e^x\right]_{-\infty}^{0} - \int_{-\infty}^{0} e^x \, dx$ parts!

$$= 1 - \left[e^x\right]_{-\infty}^{0}$$

$e^{-\infty} = 0$

$$= 1 - 1 = 0$$

(iii) $\int \frac{3x+2}{x^2+4x-5} dx$ partial fractions!

Integrand $= \frac{3x+2}{x^2+4x-5} = \frac{3x+2}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$

$$= \frac{A(x-1)+B(x+5)}{(x+5)(x-1)}$$

$\Rightarrow 3x+2 = A(x-1)+B(x+5)$

$= Ax - A + Bx + 5B$

$\left.\begin{array}{ll} [x]: & 3 = A+B \\ [1]: & 2 = -A+5B \end{array}\right\}$ add: $5 = 6B \Rightarrow B = \frac{5}{6} \Rightarrow A = \frac{13}{6}$

$\Rightarrow$ integral is $\int \left(\frac{13/6}{x+5} + \frac{5/6}{x-1}\right) dx$

$$= \frac{13}{6}\ln(x+5) + \frac{5}{6}\ln(x-1) + c$$

(b) $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$.

Substitute as suggested

$x = \tan\theta$

$\Rightarrow\ dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/4} \frac{\tan^2\theta}{(1+\tan^2\theta)^2} \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\tan^2\theta \sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

$\Rightarrow\ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \frac{1}{2}\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

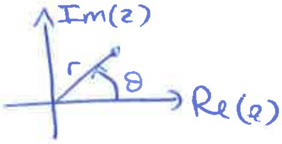
$$= \frac{\pi}{8} - \frac{1}{4}.$$

4. (a) • if $z \in \mathbb{C}$, $z = x+iy$, its complex conjugate $\bar{z} = x - iy$.

• its modulus is $\sqrt{x^2+y^2}$, the length of the line connecting z and the origin.

• its argument is the anticlockwise angle the line above makes with the positive real axis.

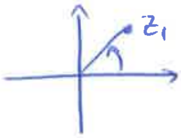
• its principal argument is the argument satisfying
$-\pi < \mathrm{Arg}(z) \leq \pi$.



(b) (i) (A) $z_1 = 2+2i$

$$|z_1| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

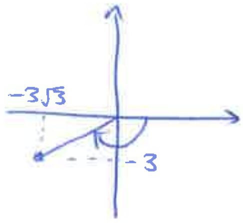
$$\mathrm{Arg}(z_1) = \pi/4$$

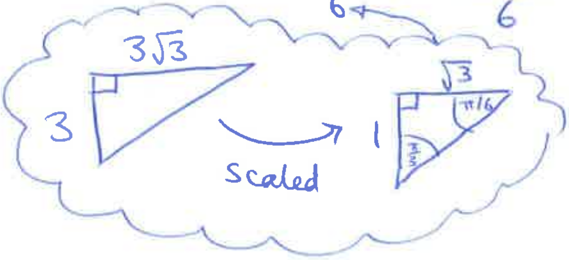


(B) $z_2 = -3\sqrt{3} - 3i$

$$|z_2| = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27+9} = \sqrt{36} = 6$$

$$\mathrm{Arg}(z_2) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}.$$





(ii) $w = \dfrac{z_1}{z_2}$ $\Rightarrow$ $w = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1-\theta_2)\right]$

$$= \frac{2\sqrt{2}}{6}\left[\cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{4}+\frac{5\pi}{6}\right)\right]$$

$$= \frac{\sqrt{2}}{3}\left[\cos\left(\frac{3\pi+10\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$$

$$= \frac{\sqrt{2}}{3}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right]$$

(iii) DeM: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

(iv) $$w^{12} = \left(\frac{\sqrt{2}}{3}\right)^{12}\left[\cos(-11\pi) + i\sin(-11\pi)\right]$$

$$= \frac{2^6}{3^{12}}\left[\cos(-\pi) + i\sin(-\pi)\right]$$

$$= \frac{64}{531{,}441}[-1] \quad \leftarrow \underline{\text{real}}!$$

obviously

(c)(i) $$\ln(z) = \ln|z| + i\arg(z)$$

$$= \ln|z| + i[\mathrm{Arg}(z) + 2\pi n], \quad n \in \mathbb{Z}$$

$$\mathrm{Ln}(z) = \ln|z| + i\mathrm{Arg}(z).$$

(ii) $$2e^{3x} = -6\sqrt{3} - 6i$$

$$\Rightarrow e^{3x} = -3\sqrt{3} - 3i$$

$$\Rightarrow 3x = \ln(-3\sqrt{3} - 3i)$$

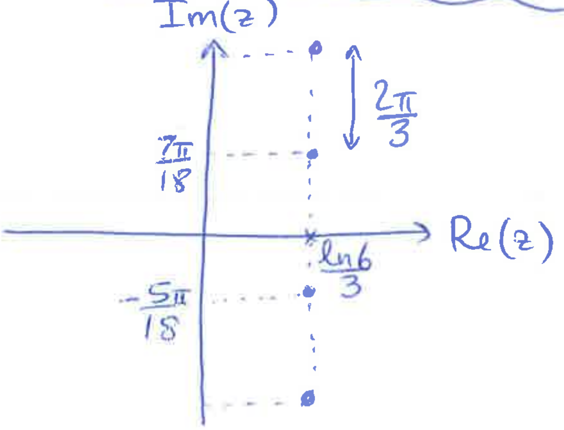
$$= \ln(z_2) \quad \leftarrow \text{from (b)(i)(B)!}$$

$$= \ln|z_2| + i[\mathrm{Arg}(z_2) + 2\pi n]$$

$$= \ln 6 + i\left[-\frac{5\pi}{6} + 2\pi n\right], \quad n \in \mathbb{Z}.$$

$$\Rightarrow \underbrace{x = \frac{\ln 6}{3} + i\left[-\frac{5\pi}{18} + \frac{2}{3}\pi n\right]}, \quad n \in \mathbb{Z}.$$

(iii)



5. (a) $\frac{1}{x+2}\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x+2}{x}$

$\Rightarrow \int \frac{1}{y}\,dy = \int\left(1+\frac{2}{x}\right)dx$

$\Rightarrow \ln y = x + 2\ln x + c$

$\Rightarrow y = e^{x+2\ln x + c}$

$= Ce^{x}e^{\ln x^2}$ Different C to c

$= Cx^2e^x$

(b) $x\frac{dy}{dx} - 2y = -x$ $\qquad y'(1)=0$

$\Rightarrow \frac{dy}{dx} - \frac{2}{x}y = -1$

$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx}\frac{1}{x^2} - \frac{2}{x^3}y = -\frac{1}{x^2}$

$\Rightarrow \frac{d}{dx}\left[y\frac{1}{x^2}\right] = -\frac{1}{x^2}$

$\Rightarrow y\frac{1}{x^2} = \frac{1}{x} + c$

$\Rightarrow \underline{y = x + cx^2}$

b.c. $y'(1)=0$: $\quad y'(x) = 1+2cx$

$0 = 1+2c \Rightarrow c = -\frac{1}{2}$

$\Rightarrow y = x - \frac{1}{2}x^2$.

(c) $y'' + y = x^2 + 2\sin x \qquad \cdots\cdots (\star)$

CF: AE: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\Rightarrow$ $y = A\sin x + B\cos x$

because $\sin x, \cos x$ in CF

PI: Try $y = Cx^2 + Dx + E + Fx\sin x + Gx\cos x$

$\Rightarrow y' = 2Cx + D + Fx\cos x + F\sin x - Gx\sin x + G\cos x$

$\Rightarrow y'' = 2C + F\cos x - Fx\sin x + F\cos x - Gx\cos x - G\sin x - G\sin x$

$= 2C + 2F\cos x - 2G\sin x - Fx\sin x - Gx\cos x$

into $(\star)$:

$$\begin{aligned} & 2C + 2F\cos x - 2G\sin x - Fx\sin x - Gx\cos x \\ & + Cx^2 + Dx + E + Fx\sin x + Gx\cos x \\ & = x^2 + 2\sin x \end{aligned}$$

$\Rightarrow$ $[x^2]$ $C = 1$

$[x]$ $D = 0$

$[1]$ $2C + E = 0 \Rightarrow E = -2$

$[\cos x]$ $2F = 0 \Rightarrow F = 0$

$[\sin x]$ $-2G = 2 \Rightarrow G = -1$

$\Rightarrow$ general solution is

$$y(x) = A\sin x + B\cos x + x^2 - 2 - x\cos x$$

6. (a) (i) f^n is odd if $f(-x) = -f(x)$

(ii) f^n is even if $f(-x) = f(x)$.

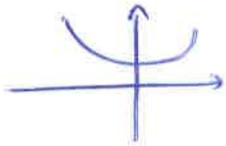
(b) $$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

(c) $f(x) = \cosh(x)$. This is an even function so $b_n = 0\ \forall n$.



Calculating coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh x\,dx = \frac{1}{2\pi}\left[\sinh x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\sinh\pi - \underbrace{\sinh(-\pi)}_{=-\sinh(\pi)}\right)$$

$$= \frac{1}{\pi}\sinh\pi$$

parts!

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\overset{\text{int}_2}{\cosh x}\,\overset{\text{diff.}}{\cos(nx)}\,dx = \frac{1}{\pi}\left\{\left[\cos(nx)\sinh x\right]_{-\pi}^{\pi} + n\int_{-\pi}^{\pi}\sin(nx)\sinh x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\cos(n\pi)\sinh(\pi) - \underbrace{\cos(-n\pi)}_{\cos n\pi}\underbrace{\sinh(-\pi)}_{-\sinh(-\pi)} + n\int_{-\pi}^{\pi}\sin(nx)\sinh x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{2\cos(n\pi)\sinh(\pi) + n\int_{-\pi}^{\pi}\overset{\text{diff}}{\sin(nx)}\,\overset{\text{int}}{\sinh x}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{2\cos(n\pi)\sinh(\pi) + n\left[\underbrace{\left[\sin(nx)\cosh x\right]_{-\pi}^{\pi}}_{0} - n\int_{-\pi}^{\pi}\cos(nx)\cosh x\,dx\right]\right\}$$

$$= \frac{1}{\pi}\left\{2\cos(n\pi)\sinh(\pi) - n^2 a_n\right\}$$

$$\Rightarrow \pi(1+n^2)\,a_n = 2\cos(n\pi)\sinh(\pi)$$

$$\Rightarrow \quad a_n = \frac{2\cos(n\pi)\sinh(\pi)}{\pi(1+n^2)}$$

$$\Rightarrow \quad f(x) = \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty} \frac{2\cos(n\pi)\sinh(\pi)}{\pi(1+n^2)} \cos(nx)$$

$$\Rightarrow \quad \cosh(x) = \frac{1}{\pi}\sinh\pi + \frac{2}{\pi}\sinh\pi \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1+n^2}\cos(nx).$$

(d) $\Rightarrow \quad \pi\coth(x) = 1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1+n^2}\cos(nx).$

Let $x = \pi$. Then

$$\pi\coth(\pi) = 1 + 2\sum_{n=1}^{\infty}\frac{\cos^2(n\pi)}{1+n^2}$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2}.$$

$$\Rightarrow \quad \frac{1}{2}\left[\pi\coth(\pi) - 1\right] = \sum_{n=1}^{\infty}\frac{1}{1+n^2}$$

$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots$$