*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define $\cosh x$ and $\sinh x$ in terms of exponentials. Use these definitions to:
   (i) obtain the formula $\cosh^2 x - \sinh^2 x = 1$,
   (ii) find the derivatives of $\cosh x$ and $\sinh x$,
   (iii) show that
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right).$$

   (b) Define $\tanh x$ and $\operatorname{sech} x$ in terms of $\cosh x$ and $\sinh x$. Use these definitions to:
   (i) find, from part (a)(i), an equation that links $\tanh^2 x$ and $\operatorname{sech}^2 x$,
   (ii) show, using the results of part (a)(ii), that
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \operatorname{sech}^2 x, \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x.$$

   (c) The curve $y(x)$ is given by
$$y(x) = 2\tanh^2 x + 5\operatorname{sech} x - 4.$$

   Find:
   (i) the two real values of $x$ where $y(x)$ cuts the $x$-axis,
   (ii) the values of $x$ at any stationary points of $y(x)$,
   (iii) the nature of these stationary points (i.e. are they minima, maxima, etc.?).

   Give your answers in terms of logarithms (as opposed to inverse hyperbolic functions) where appropriate.

2. (a) (i) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}.$$

(ii) Find, or simply write down, the first three terms of the Maclaurin series of $(1-x)^\alpha$, for some real constant $\alpha$.

(iii) Hence write down the first three terms of the Maclaurin series of $(1-x^2)^\alpha$.

(iv) Using parts (i) and (iii), or otherwise, write down the first three terms of the Maclaurin series of $\sin^{-1}(x)$.

(b) The *Helmholtz equation* is an important partial differential equation which arises in many physical problems in (amongst other things) electromagnetism and acoustics. It is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \alpha\psi$$

for a constant $\alpha$.

If $\alpha = -6$, determine whether the following functions $\psi(x, y, z)$ satisfy the Helmholtz equation:

(i) $\psi(x, y, z) = \dfrac{1}{2}\sin(-2y)\cos(-z)[\sin(x) + \cos(x)]$,

(ii) $\psi(x, y, z) = \dfrac{e^{x/3}}{yz}$, (given $y, z \neq 0$).

3. (a) Evaluate the integral

$$\int \frac{1}{\sqrt{1-x}}\,\mathrm{d}x.$$

(b) Show that

$$\int_0^1 \sin^{-1} x\,\mathrm{d}x = \frac{\pi}{2} - 1.$$

*Hint:* you might want to use both substitution and integration by parts, in some order.

(c) Show that

$$\int_1^\infty \frac{x+3}{x(x+2)^2}\,\mathrm{d}x = \frac{3}{4}\ln(3) - \frac{1}{6}.$$

4. (a) Let $z = -3 + 4i$ and $w = -i$. Plot $z$ and $w$ on an Argand diagram.
Then write down:
   (i) $\arg(z - \overline{w})$,
   (ii) $|z/w|$,
   (iii) $\operatorname{Im}(zw)$,
   (iv) $w^{88291}$,
   (v) $\ln(w)$,

   where your answers to (iv) and (v) are in the form $x + iy$, where $x, y \in \mathbb{R}$.

   (b) Find all solutions to the equation

$$z^3 = -8,$$

   writing your answers in either Cartesian or modulus–argument form.

   (c) Let $z = \cos\theta + i\sin\theta$.
   (i) State De Moivre's theorem.
   (ii) By considering $z^4$, show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

5. (a) Solve the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}e^{x+y}.$$

   (b) Solve the first order differential equation

$$\cot x \frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos x \cot x,$$

   with the boundary condition $y(0) = 1$.

   (c) Solve the second order differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x.$$

   *Hint:* use the substitution $x = e^t$.

6. (a) Explain what it means for a function to be *odd* or *even*, and give an example of:

    (i) a function which is odd,

    (ii) a function which is even,

    (iii) a function which is both odd and even.

   (b) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (c) The function $f(x)$ is given by

$$f(x) = \begin{cases} \pi + x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 \leqslant x < \pi \end{cases},$$

   where $f(x + 2\pi) = f(x)$.

    (i) Sketch the function in the region $-2\pi < x < 2\pi$.

    (ii) Show that the Fourier series for $f(x)$ is given by

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25\pi}\cos 5x + \cdots .$$

   (d) Using part (c)(ii), or otherwise, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$