The fluid dynamics of chocolate fountains
(synopsis)

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14 July 2012

Abstract

Chocolate fountains are popular features at special events. But why do they never use white chocolate? Would they work with water? And why, when the chocolate falls, does it fall slightly inwards? In this most delectable of studies, we investigate the trade-off between accuracy and simplicity in models used by commercial chocolatiers. In different geometries of the fountain, we solve the governing equations—mostly analytically—and compare these results with observations from our own fountain experiments. We find that, with some limitations, our models are in fact good for a number of non-Newtonian fluid problems. Marshmallows are, sadly, not provided.

1 Modelling chocolate

One of the first problems when modelling chocolate is that there is no ‘standard’ chocolate: even ignoring brand differences, the proportions of cocoa solids and cocoa butter vary greatly: white chocolate, for example, contains no cocoa solids at all. How a fluid behaves—for example, chocolate for fountains should run smoothly and consistently—is largely down to its viscosity. Higher proportions of cocoa butter make chocolate more viscous, which is why you don’t see many white chocolate fountains.

If, for now, we assume that we *are* dealing with some ‘standard’ chocolate, we can look at some properties of its viscosity. Temperature plays a role: as we heat chocolate, it melts and becomes less viscous. Once melted, shearing chocolate (for example, smearing melted chocolate on top of a cake) will also decrease its viscosity. In our chocolate fountain the temperature is actually very stable at $40\,^{\circ}\mathrm{C}$, and it is the rate of shear which plays the greatest role in determining the viscosity of the chocolate.

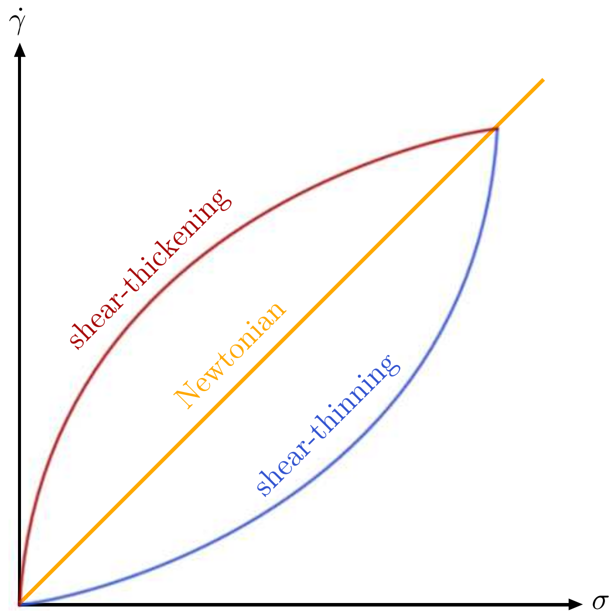

Figure 1: Graph of rate of strain $\dot{\gamma}$ against stress σ for a typical shear-thickening (red), Newtonian (orange) and shear-thinning (blue) fluid.

We might assume that a model which gives chocolate no viscosity at all (an *inviscid* model), modelling it like air or water, would match poorly with observation. Although that is true while the chocolate is in contact with the fountain, as it falls freely we find that it is not a terrible approximation, and it exhibits some of the observed behaviour.

When we look for a viscous model, because of the importance of shear rate on its viscosity, chocolate is in a class of fluids known as *shear-thinning*. Along with paint and blood, chocolate's viscosity decreases in response to an increase in its shear rate. Paint needs to have this quality so that it can run easily onto the wall when it's being applied, but not run when it is left to dry. Similarly, blood should clot if it's not moving. Shear-thinning fluids are more complicated to model than *Newtonian* liquids such as milk, whose viscosities remain constant at different shear rates. Fluids also exist that do the opposite to shear-thinning: their viscosities *increase* in response to an increase in shear rate. These are known as *shear-thickening*, and the best example is a paste of cornflour and water.

What these latter three different types of fluid have which defines them is a different relationship between their viscosities and the shear rate applied to them. We normally introduce stress at this point, calling viscosity the ratio of stress and strain rate (hence the gradient of their graph), and call the relation between stress and strain rate of a fluid the *constitutive relation*. Figure 1 illustrates the differences graphically.

In this project we ask what the most suitable model is for the type of chocolate we have in our table chocolate fountain. In order to do this, we simplify our model, illustrated in Figure 2. We look at three geometries: the pipe, the dome and the

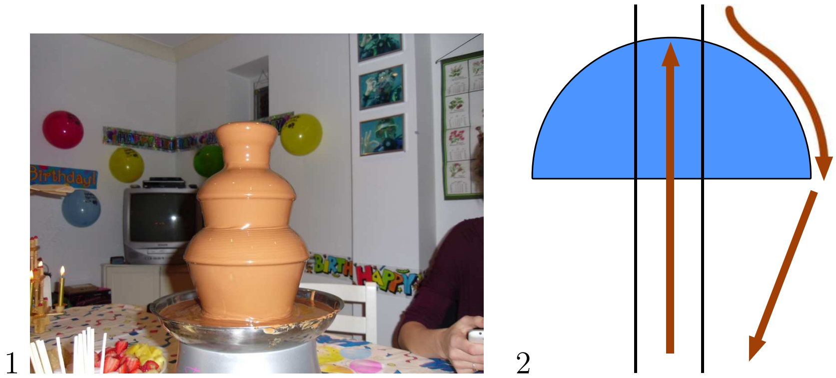

Figure 2: We model the chocolate fountain (1) as three geometries (2).

falling sheet, and compare our predictions to experimental results.

In order to compare our predictions with our observations, we need experimentally-found values of constants which appear in the models. Fortunately there is sufficient published work on chocolate from the confectionery industry which allows us to use appropriate values. We use these data to fix all the parameters in our models: there are no free parameters here to artificially improve our fit to the experiments.

2 The pipe

In our table chocolate fountain, the melted chocolate is transported from the bottom basin, where it is heated, to the top by use of an Archimedes screw. The mechanics of the fluid transport in the screw are outside the scope of the project, since we cannot observe the chocolate in the pipe, and the rotation which it produces at the top is quickly dissipated as the chocolate travels over the dome. We focus instead on a vertical pipe with a constant pressure gradient (in fact on a larger fountain with a pump this would be a better physical approximation).

The governing equations for this incompressible fluid flow are the Navier–Stokes equations,

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \mathbf{F}$$

where $\boldsymbol{\sigma}$ is the stress tensor, ρ is density, $\mathbf{u}$ is fluid speed, p is pressure, $\mathbf{F}$ are external forces (per unit volume), and $\mathrm{D}/\mathrm{D}t$ is the material derivative; and the continuity equation,

$$\nabla\cdot\mathbf{u} = 0.$$

We will now offer three isothermal constitutive relations as candidates.

The first is the Newtonian viscous fluid. Although it looks to be a poor choice for modelling chocolate since it assumes a constant viscosity, its compared analytical simplicity to the alternatives makes it one we should consider. This relation is given (in scalar form) by

$$\sigma = \mu\dot{\gamma},$$

where σ is stress, μ is the constant viscosity and $\dot{\gamma}$ is the rate of strain. The resulting velocity profile is parabolic, shown in Figure 3.1. Note that the figure is drawn with zero pressure gradient but in the presence of gravity, so we see downward velocity.

In 1959, N. Casson provided a model, originally designed for printing ink, which gives a constitutive equation of

$$\sqrt{\sigma} = \begin{cases} \sqrt{\mu_{\mathrm{C}}\dot{\gamma}} + \sqrt{\sigma_{\mathrm{y}}} & \text{if } \sigma > \sigma_{\mathrm{y}} \\ \sqrt{\sigma_{\mathrm{y}}} & \text{if } \sigma \leq \sigma_{\mathrm{y}} \end{cases},$$

with two physical constants, μ_{C} and σ_{y}.

Casson's model was recommended by the International Confectionery Association to model chocolate from 1973 until 2000, when Aeschlimann and Beckett (2000) found that at lower shear rates, this model does not match experimental data well. It was difficult to find repeatability, and so interpolation data has been recommended since.

Interpolation data typically takes the form of a *power-law* model, given by

$$\sigma = k\dot{\gamma}^n,$$

for a given constant k. A typical value of n for chocolate is $n = 1/3$. Fluids with $n < 1$ are shear-thinning, and those with $n > 1$ are shear-thickening. Newtonian fluids have $n = 1$.

Using our physical values of n and k, we find that the power-law profile (Figure 3.2) is considerably flatter and slower than the Newtonian one (Figure 3.1): we get plug flow (almost) in the centre of the pipe. Casson's model, with appropriate constants, produces a graph (Figure 3.3) which is very similar to the Newtonian model, since the plug flow radius turns out to be very small compared to the pipe radius. Artificially increasing the value of the plug flow radius (Figure 3.4) would make the velocity profile more like the power-law model, but we are not at liberty to do this.

3 The dome

For the fluid travelling on the dome, we limit our models to just two: Newtonian and power-law. We expect the profile on the dome to be similar to half a pipe, and Casson's model with experimental data did not significantly differ from these two in the pipe. Using the convenience that $n = 1/3$ in the power-law model for chocolate, we proceed to solve the governing equations of these two models analytically within the geometry of the dome, and importantly make use of the scaling that the thickness of the chocolate is considerably smaller than the radius of the dome.

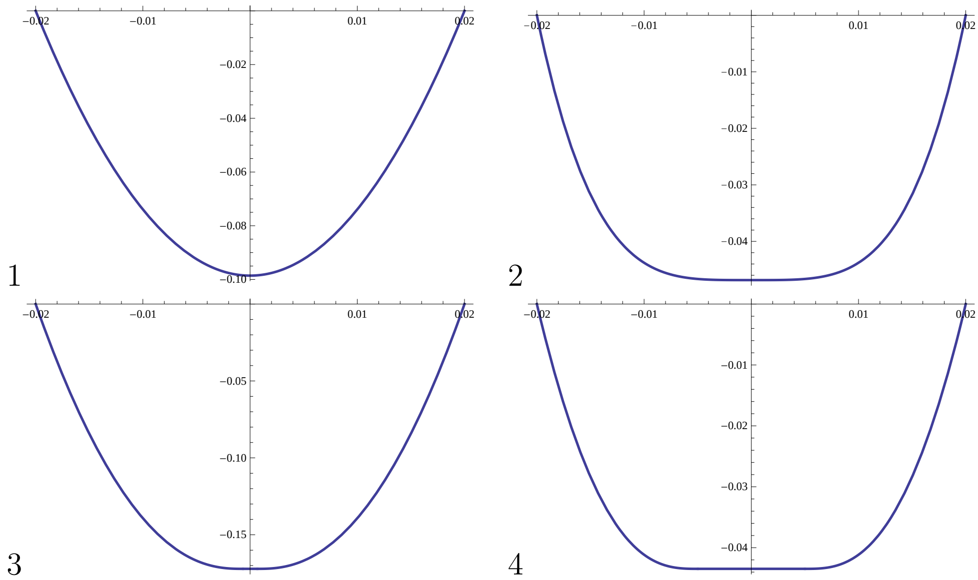

Figure 3: Velocity profiles of pipe flows in gravity with zero pressure gradient: 1. Newtonian fluid. 2. (Shear-thinning) power-law fluid. 3. Casson fluid. 4. Casson fluid with exaggerated plug flow radius.

After this scaling, we find that the curvature of the dome disappears: the problem in both cases reduces to the fundamental equations of thin-film flow, being a pure balance of viscosity and gravity. This shouldn't be altogether surprising: our film thickness changes slowly over the dome and is tiny compared to the dome's radius. Furthermore the speeds we experience on the dome are not particularly fast. We can draw parallels with slow lava flows or problems in geophysical fluids.

We can find, then, velocities and thickness of the chocolate on the dome as a function of angle, as in Figure 4. In comparing the Newtonian and power-law cases at different film thicknesses, we find that the Newtonian fluid is on average faster (matching the pipe), except close to the surface of the dome. Figure 5 shows that the Newtonian model predicts thinner films than the chocolatey power-law model, which is what we expect from faster flow. However, these films are both predicted to be much thicker than seen in experiment. The global thickness is ultimately governed by the input flux, which we found by estimating flow speed on the dome. If the no-slip condition was not being satisfied on the dome in practice, this would result in higher speeds being observed than we predicted, causing us to overestimate the flux. Our data for constants comes from a few different sources, so this could also introduce inaccuracies.

We now reconsider temperature. As chocolate cools, it becomes more viscous and this could be a problem as it falls down the dome. However, when measured, the temperature of the chocolate everywhere was $40 \pm 0.5\,°\mathrm{C}$. Wollny (2005) suggests that a change of one degree causes at most a 10% change in viscosity, which is not very significant in our table chocolate fountain. For completeness (e.g. in larger fountains) we did look at temperature-dependent viscosity: the results were as Leslie et al. (2011).

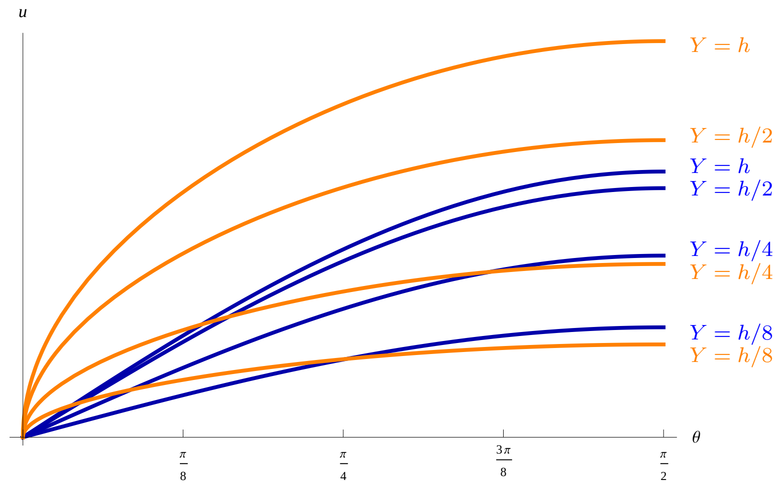

Figure 4: Velocities following streamlines at different film thicknesses, as functions of the angle from the top, θ. Newtonian in orange, chocolatey power-law fluid in blue. Y measures distance from the surface of the dome, $Y = h$ represents the surface of the fluid. Same flux as in Figure 5.

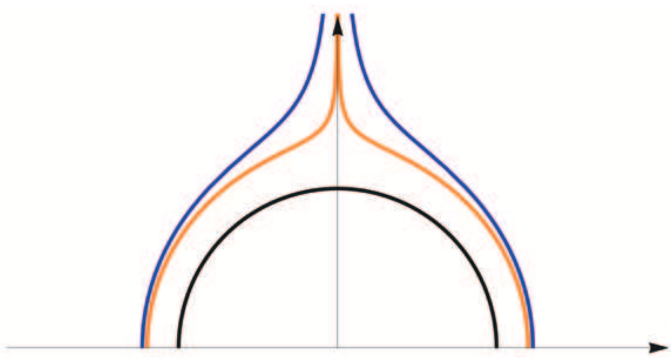

Figure 5: Flows thicknesses on the dome in the Newtonian (orange) and power-law (blue) cases.

4 The sheet

The falling sheet is the most impressive part of the chocolate fountain, but also the most mathematically difficult! The reason for this is threefold: there are two free boundaries (each side), the relation between the sheet thickness and distance along the sheet is unknown, and the position of the sheet's trajectory in space is also unknown.

We concern ourselves with steady flows, although considerable investigation could be made into the unsteady nature of the falling sheet, and we start with the simplest, inviscid fluid model. Extensive work, starting with Taylor and Howarth (1959), has been conducted on water bells, where a jet of water is impacted onto a flat surface, and the water spreads out in a thin sheet (the water-hitting-a-spoon effect!). Surface tension then pulls the sheet back inwards to form a closed water bell: the same inward-falling effect as observed in our chocolate fountain.

We use the scaling that the sheet is very thin compared to its height, and so velocity is constant throughout the cross-section (this is where we need the inviscid approximation). To find an analytical solution, we further assume that the radius of the dome is considerably smaller than the height of the sheet. Although this is true for water bells, this assumption is not good for *us* to make, since these are of similar order. However, doing so allows us to predict the trajectory of the sheet (Figure 6, in blue), which falls inwards even though the fluid is inviscid, although more so than we see in experiment (in brown). Energy arguments also invalidate the inviscid assumption.

Gilio et al. (2005) derives equations of motion for inviscid and Newtonian viscous falling sheets, with a constant viscosity in the Newtonian case very similar to chocolate under this shear. We match our inviscid prediction from the work above with theirs to plot the predicted path of the sheet (Figure 6, in pink) which agrees pretty well with experiment (in brown). Artificially altering the surface tension parameter would generate falling sheets with steeper/shallower angles, although we are not at liberty to do this. Note that there is considerable difference at the top of the sheet between predictions and reality: we attribute this to the teapot effect, caused by atmospheric pressure and wetness of the base of the dome.

5 Conclusion

We started with four potential models. Analysis in the pipe allows us to ignore Casson's model on the dome, where we find a power-law model which we can solve analytically to be the most appropriate. We also find that scaling removes the curvature of the problem, reducing it to thin-film flow. In the sheet, we find that variations in viscosity are not as important as the presence of viscosity itself, and we have been able to find a good match to experiment with a Newtonian viscous

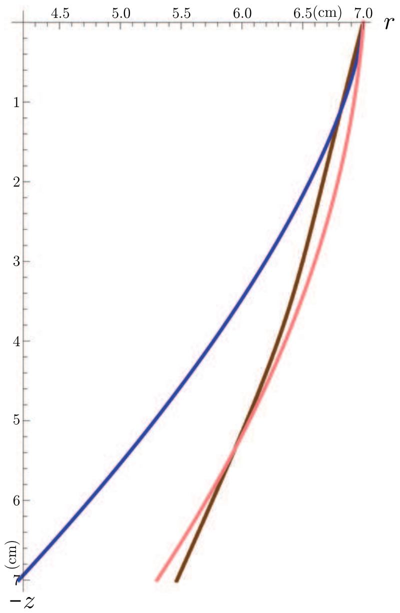

Figure 6: Profile of the falling sheet in the inviscid (blue) and viscous (pink) cases. Experimental observation (brown) included for comparison.

model, where all our parameters come from commercial experiments. Further work into instabilities in the falling sheet is critical to coating flows, not just in chocolate but also in industrial settings like the manufacture of technology such as television screens.

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